To return to the upper and lower variational bounds on the K matrix elements, we feel that our approach will have applications in many unsolvable problems—that is, those problems which cannot be solved on a finite computer. Among these applications are atomic and molecular collisions, models for nuclear reactions, many-body problems, the relativistic Bethe-Salpeter equation, and to dispersion relations. We plan to pursue some of these problems later.

Finally, we would like to stress the importance of the pioneering work of Spruch and his collaborators on variational principles. Even though our upper and lower bounds are quite distinct and independent, and bear little relation to the lower bound principle of Spruch, his work suggested several points which were included in our discussion.

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Theorem on Separation of Variables with Application to Static-Source Meson Theory

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A general formulation is given of the process by means of which one eliminates certain redundant degrees of freedom occurring in interacting quantum-mechanical systems. This may be considered a generalization of the transformation to center of mass in problems having translational invariance. When applied to the pion-nucleon system, the static-source approximation arises as the first term in a series, the higher terms of which are easily calculable using an algorithm developed in the Appendix. Thes system is not an expansion in the inverse mass of the nucleon, and it is shown that the static source approximation need not be considered as a nonrelativistic approximation.

INTRODUCTION

IN energy eigenvalue problems involving interacting systems and possessing translational invariance, one usually transforms to center-of-mass coordinates, eliminates redundant degrees of freedom, and consequently obtains an equivalent problem in a reduced Hilbert space. It is not surprising that such a reducibility exists whenever interacting systems possess an Abelian invariance group. A general procedure by which such reductions may be effected is discussed in Sec. II. Section III contains a heuristic example.

Of some interest may be the simple field-theoretic examples in Sec. IV and V. It is demonstrated, using a hypothetical heavy boson-light boson interacting system and the nucleon-pion system, how such interacting-field theories lead straightforwardly to an approximation expansion the first term of which is a "static source" theory. No statements about "nonrelativistic approximations" are made. Succeeding terms in the series are easily obtained using an algorithm developed in the Appendix.

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II. ELIMINATION OF DEGREES OF FREEDOM ASSOCIATED WITH CONSERVED ADDITIVE QUANTUM NUMBERS

Consider an energy eigenvalue problem in a Hilbert space which can be decomposed into the form

$$\mathfrak{H}=\mathfrak{H}_j\otimes\tilde{\mathfrak{H}}. \tag{1}$$

Let H be the Hamiltonian of the system and let

$$J=j+\tilde{j};\quad J=\{J_1, J_2, \cdots\} \tag{2}$$

be a set of operators such that

$$[H,J]=0 \tag{3}$$

and such that the j are a complete commuting set for $\mathfrak{H}_j$. Let some appropriate complete commuting set for $\tilde{\mathfrak{H}}$ be denoted by k. One can then use the following notation[1]:

$$|j'\rangle\epsilon\mathfrak{H}_j;\quad |k'\rangle\epsilon\tilde{\mathfrak{H}};\quad |j',k'\rangle\epsilon\mathfrak{H}. \tag{4}$$

Let $|\psi\rangle$ be the simultaneous eigenstate in $\mathfrak{H}$ of H and the J,

$$H|\psi\rangle\equiv H'|\psi\rangle;\quad J|\psi\rangle\equiv J'|\psi\rangle\,. \tag{5}$$

[1] Primed symbols will always denote eigenvalues of corresponding operators.

We can now decompose $|\psi\rangle$ as[2]

$$|\psi\rangle \equiv \sum_{j'} |j'\rangle |\psi,j'\rangle . \tag{6}$$

Inserting (2) and (5) into (6), one obtains

$$\tilde{j}|\psi,j'\rangle = (J'-j')|\psi,j'\rangle . \tag{7}$$

Since the $|\psi,j'\rangle$ are eigenstates of the $\tilde{j}$ in $\tilde{\mathcal{H}}$ and have different eigenvalues for different j', then

$$\langle\psi,j''|\psi,j'\rangle = \delta_{j'j''}\langle\psi,j'|\psi,j'\rangle . \tag{8}$$

If we normalize

$$\langle\psi|\psi\rangle \equiv 1, \quad \langle j'|j''\rangle \equiv \delta_{j'j''}, \tag{9}$$

then

$$\sum_{j'} \langle\psi,j'|\psi,j'\rangle = 1. \tag{10}$$

Now we define

$$|\psi:J'\rangle \equiv \sum_{j'} |\psi,j'\rangle \tag{11}$$

so that

$$\langle\psi:J'|\psi:J'\rangle = 1. \tag{12}$$

The purpose of this section is to show that there always exists an equation for $|\psi:J'\rangle$ in the reduced $\tilde{\mathcal{H}}$ which is equivalent to the original problem, Eq. (5).

In Eqs. (6), (10), and (11) the $\sum_{j'}$ extends over the range of eigenvalues of j in $\mathcal{H}_j$, consequently, in general, the values taken on by the function $J'-j'$ may not extend over the range of eigenvalues of $\tilde{j}$ in $\tilde{\mathcal{H}}$ except in particular cases such as when the range eigenvalues of j extends from $-\infty$ to $+\infty$. In such cases one can write

$$\sum_{j'} \tilde{P}_{J'-j'} = \tilde{1}, \tag{13}$$

where $\tilde{P}_{j'}$ is the projection operator in $\tilde{\mathcal{H}}$ onto eigenstates of eigenvalue j', and $\tilde{1}$ is the unit operator in $\tilde{\mathcal{H}}$. In any case, however, the fact that $|\psi\rangle$ satisfies Eqs. (5) and consequently (7) implies that

$$\tilde{P}_{J'-j'}|\psi:J'\rangle = |\psi,j'\rangle, \tag{14}$$

$$\sum_{j'} \tilde{P}_{J'-j'}|\psi:J'\rangle = \tilde{1}|\psi:J'\rangle = |\psi:J'\rangle. \tag{15}$$

As a consequence of the foregoing, Eqs. (5) may be written as

$$\sum_{j'} H|j'\rangle\tilde{P}_{J'-j'}|\psi:J'\rangle = \sum_{j''} H'|j''\rangle\tilde{P}_{J'-j''}|\psi:J'\rangle. \tag{16}$$

Equation (16) is the starting point for two somewhat different ways to proceed. Consider first the following conditions in $\tilde{\mathcal{H}}$ implied by (16):

$$\tilde{P}_{J'-j''}\sum_{j'}\langle j''|H|j'\rangle\tilde{P}_{J'-j'}|\psi:J'\rangle = \tilde{P}_{J'-j''}H'|\psi:J'\rangle$$

(for every j'' in the eigenvalue range of j). (17)

[2] Σ includes possible integration.

Now $\langle j''|H|j'\rangle$ is an operator in $\tilde{\mathcal{H}}$ parametrically dependent on j'' and j' can be expressed generally as

$$\langle j''|H|j'\rangle = \sum_{mn} h_m(j'')H_{mn}h_n(j'), \tag{18}$$

where h_n are c-number functions and H_{mn} operators in $\tilde{\mathcal{H}}$. The determination by (18) of the h_n is incomplete when the range of the j' is finite since (18) is a condition only for j' within this range. For any h_n satisfying (18) we define

$$\tilde{H} \equiv \sum_{mn} h_m(J'-\tilde{j})H_{mn}h_n(J'-\tilde{j}) \tag{19}$$

with which from (17) one obtains

$$\tilde{P}_{J'-j''}\tilde{H}\sum_{j'}\tilde{P}_{J'-j'}|\psi:J'\rangle = \tilde{P}_{J'-j''}H'|\psi:J'\rangle$$

(for every j'' in the eigenvalue range of j). (20)

Using Eq. (15) this implies

$$\tilde{P}_{J'-j''}\tilde{H}|\psi:J'\rangle = \tilde{P}_{J'-j''}H'|\psi:J'\rangle . \tag{21}$$

(for every j'' etc.) (21)

Now, if condition (13) is satisfied, we can write immediately

$$\tilde{H}|\psi:J'\rangle = H'|\psi:J'\rangle \tag{22}$$

for any $\tilde{H}$ constructed such that (18) is satisfied, and, in particular, the simplest such (e.g., using lowest order polynomial form in j' for the h_n). Otherwise we shall have to take care to rig the h_n such that

$$h_n(j') \doteq 0, \tag{23}$$

where the conditional mark over the equal sign means it holds only for j' *outside* the range of eigenvalues of j and $J'-j$ *inside* the range of eigenvalue of $\tilde{j}$.

Condition (23) may require an awkward form for $\tilde{H}$. If so, there is another possible treatment of (16), leading to a possibly less complicated form for $\tilde{H}$. Define

$$|\theta\rangle \equiv \sum_{j'} |j'\rangle \epsilon\tilde{H}_j \tag{24}$$

and multiply (16) from the left by its conjugate bar, yielding

$$\sum_{j'}\langle\theta|H|j'\rangle\tilde{P}_{J'-j'}|\psi:J'\rangle = H'|\psi:J'\rangle. \tag{25}$$

Proceeding as before,

$$\langle\theta|H|j'\rangle \equiv \sum_{n} H_n h_n(j'), \tag{18'}$$

$$\sum_{n} H_n h_n(J'-\tilde{j}) \equiv \tilde{H}, \tag{19'}$$

and with this new definition, (22) again holds. Note that in this case (18') need only hold in the range of eigenvalues of j. Note also that now $\tilde{H}$ need not be Hermitian.

Our conclusion is then for every solution of (5) (the problem in the full Hilbert space) there exists a solution of (22) (a problem in a reduced Hilbert space). Conversely, every solution of (22) satisfying also (15) is a solution to (5). Consequently the knowledge of all solutions of (22) is equivalent to the knowledge of all solutions of (5).

III. A SIMPLE EXAMPLE USING A SPIN-SPIN INTERACTION

To illustrate the method, consider two systems of spin $\frac{1}{2}$ governed by a Hamiltonian of the form

$$H=\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)}, \tag{26}$$

where

$$S_1=\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad S_2=\frac{1}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad S_3=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Of course, in this simple case we know all the answers beforehand, namely, that H has eigenvalues,

$$H'=-\tfrac{3}{4}\quad\text{or}\quad+\tfrac{1}{4}$$

corresponding to total spin 0 or 1, respectively. In this case the conserved additive quantum number is

$$S_3=S_3{}^{(1)}+S_3{}^{(2)} \tag{2'}$$

and we know its possible eigenvalues S_3'. The Hilbert space of this problem decomposes naturally into

$$\mathfrak{H}=\mathfrak{H}^{(1)}\otimes\mathfrak{H}^{(2)} \tag{1'}$$

$$|S_3{}^{(1)\prime}\rangle\epsilon\mathfrak{H}^{(1)};\quad |S_3{}^{(2)\prime}\rangle\epsilon\mathfrak{H}^{(2)};\quad |S_3{}^{(1)\prime},S_3{}^{(2)\prime}\rangle\epsilon\mathfrak{H} \tag{4'}$$

and we shall reduce the problem to one in $\mathfrak{H}^{(2)}$ only

$$\tilde{\mathfrak{H}}\rightarrow\mathfrak{H}^{(2)}. \tag{27}$$

Let

$$S_3{}^{(1)\prime}\equiv j';\quad \mathbf{S}^{(2)}\equiv\tfrac{1}{2}\boldsymbol{\sigma}.$$

Then one easily obtains

$$\begin{aligned}\langle j''|H|j'\rangle=\tfrac{1}{4}[&\delta_{j'',\frac{1}{2}}\delta_{j',\frac{1}{2}}\sigma_3\\ &-\delta_{j'',-\frac{1}{2}}\delta_{j',-\frac{1}{2}}\sigma_3+\delta_{j'',\frac{1}{2}}\delta_{j',-\frac{1}{2}}(\sigma_1-i\sigma_2)\\ &+\delta_{j'',-\frac{1}{2}}\delta_{j',+\frac{1}{2}}(\sigma_1+i\sigma_2)].\end{aligned} \tag{28}$$

One may use as a representation of the Kronecker δ's

$$\delta_{j',\pm\frac{1}{2}}=(\tfrac{1}{2}\pm j'), \tag{29}$$

which gives for $S_3'=0$,

$$\tilde{H}=\begin{pmatrix}-\frac{1}{4}, & \frac{1}{2}\\ \frac{1}{2}, & -\frac{1}{4}\end{pmatrix} \tag{30}$$

using the form defined by (19). Solving the eigenvalue equation for this $\tilde{H}$ one gets indeed the correct eigenvalues and Clebsch-Gordan coefficients.[3] In the case $S_3'=\pm1$, one should use the definition of $\tilde{\mathfrak{H}}$ given by (19') which results in,

$$\tilde{H}=\begin{cases}\frac{1}{4}\begin{pmatrix}1 & 4\\ 0 & -3\end{pmatrix} & \text{when}\quad S_3'=1\\ \frac{1}{4}\begin{pmatrix}-3 & 0\\ 4 & 1\end{pmatrix} & \text{when}\quad S_3'=-1.\end{cases} \tag{31}$$

In each case only one eigenfunction, the correct one corresponding to eigenvalue $H'=\frac{1}{4}$, satisfies (15) qualifying as a true solution. The other solution is a kind of "ghost."

IV. EXAMPLE OF INTERACTING SCALAR BOSONS LEADING TO A STATIC-SOURCE MESON THEORY

We consider two scalar boson fields, a light field $A(x)$ of mass μ and a heavy field $B(x)$ of mass m. The Hamiltonian is taken to be

$$\begin{aligned}&H=H_{0A}+H_{0B}+I\\ &H_{0A}=\sum_{\mathbf{k}}\omega_{\mathbf{k}}a_{\mathbf{k}}{}^{*}a_{\mathbf{k}};\quad \omega_{\mathbf{k}}{}^{2}=\mathbf{k}^2+\mu^2\\ &H_{0B}=\sum_{\mathbf{k}}\epsilon_{\mathbf{k}}b_{\mathbf{k}}{}^{*}b_{\mathbf{k}};\quad \epsilon_{\mathbf{k}}{}^{2}=\mathbf{k}^2+m^2\\ &I=g\int:B^2(x):A(x)d\mathbf{x}.\end{aligned} \tag{32}$$

It is assumed that there exists energy eigenstates of the system and that some of these are describable as low-lying one-particle states. This means that eigenstates exist having mainly one bare B quantum. One can then approximate[4] the situation by cutting off the Hilbert space beyond the one-B quantum subspace. Symbolically expressed,

$$\mathfrak{H}=\mathfrak{H}_B\otimes\mathfrak{H}_A, \tag{33}$$

$$\mathfrak{H}_B=\mathfrak{H}_{0B}\oplus\mathfrak{H}_{1B}\oplus\cdots\approx\mathfrak{H}_{1B}, \tag{34}$$

under which restriction[5]

$$H_{0B}\approx\epsilon\equiv(p^2+m^2)^{1/2} \tag{35}$$

$$\begin{aligned}I\approx g\int(d\mathbf{x}/V)A(x)\epsilon^{-1/2}e^{i\mathbf{p}\cdot\mathbf{x}}\\ \times\sum_{\mathbf{p}'\mathbf{p}''}|\mathbf{p}'\rangle\langle\mathbf{p}''|\epsilon^{-1/2}e^{i\mathbf{p}\cdot\mathbf{x}},\end{aligned} \tag{36}$$

[3] This method may actually be useful in solving for the coupling coefficients for, e.g., representations of groups other than the rotation group.

[4] This no-pairs approximation is, of course, not necessary. One could keep more of the heavy-particle Hilbert space for a more refined theory and still "reduce" the problem because of its translational invariance.

[5] It is interesting to note that

$$\sum_{\mathbf{p}'\mathbf{p}''}|\mathbf{p}'\rangle\langle\mathbf{p}''|$$

can be interpreted as the projection operator onto states in which the B particle is at the origin.

where

$$|\mathbf{p}'\rangle\epsilon\mathfrak{C}_{1B};\quad \mathbf{p}|\mathbf{p}'\rangle\equiv\mathbf{p}'|\mathbf{p}'\rangle,$$
$$\epsilon|\mathbf{p}'\rangle=(\mathbf{p}'^2+m^2)^{\frac{1}{2}}|\mathbf{p}'\rangle\equiv\epsilon'|\mathbf{p}'\rangle. \tag{37}$$

Calling $\mathbf{P}_A$ the total momentum operator of the A field we have the three additive conservation laws

$$[\mathbf{p}+\mathbf{P}_A, H]=0 \tag{3'}$$

and can thus proceed to use the general results of Sec. II. Following the rules outlined by Eq. (19), one obtains, in the case in which the total momentum of the system is zero,[6]

$$\tilde{H}=H_{0A}+(\mathbf{P}_A{}^2+m^2)^{\frac{1}{2}}+g(\mathbf{P}_A{}^2+m^2)^{-\frac{1}{4}}A(0)(\mathbf{P}_A{}^2+m^2)^{-\frac{1}{4}}, \tag{38}$$

where

$$A(0)\equiv V^{-1/2}\sum_{\mathbf{k}}(2\omega_{\mathbf{k}})^{-1/2}(a_{\mathbf{k}}{}^*+a_{\mathbf{k}}), \tag{39}$$

$$\mathbf{P}_A=\sum_{\mathbf{k}}\mathbf{k}a_k{}^*a_{\mathbf{k}}. \tag{40}$$

The solutions of

$$(\tilde{H}-H')|\psi,0\rangle;\quad |\psi,0\rangle\epsilon\mathfrak{C}_A=\tilde{\mathfrak{C}} \tag{21'}$$

are in one-to-one correspondence with those solutions of the original problem in $\mathfrak{C}_{1B}\otimes\mathfrak{C}_A$ having zero total momentum and energy $H'\cdot|\psi,0\rangle$ is the state of the A field when after the total system is prepared in the energy eigenstate $|\psi\rangle$, the particle's position is measured and found to be at the coordinate origin.

The effective Hamiltonian can be expanded in normal ordered form using the general algorithm developed in the Appendix.

$$(\mathbf{P}_A{}^2+m^2)^{1/2}=m+\sum_{\mathbf{k}}(\epsilon_{\mathbf{k}}-m)a_{\mathbf{k}}{}^*a_{\mathbf{k}}+\tfrac{1}{2}\sum_{\mathbf{k}\mathbf{k}'}[\epsilon_{\mathbf{k}+\mathbf{k}'}-\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}'}+m]a_{\mathbf{k}}{}^*a_{\mathbf{k}'}{}^*\cdot a_{\mathbf{k}}a_{\mathbf{k}'}+\cdots \tag{41}$$

$$(\mathbf{P}_A{}^2+m^2)^{-1/2}a_{\mathbf{k}}{}^*(\mathbf{P}_A{}^2+m^2)^{-1/2}=(\epsilon_{\mathbf{k}}m)^{-1}a_{\mathbf{k}}{}^*+\sum_{\mathbf{k}'}[(\epsilon_{\mathbf{k}+\mathbf{k}'}\epsilon_{\mathbf{k}'})^{-1}-(\epsilon_{\mathbf{k}}m)^{-1}]a_{\mathbf{k}}{}^*a_{\mathbf{k}'}{}^*\cdot a_{\mathbf{k}'}+\cdots. \tag{42}$$

Inserting these into (38) yields

$$\tilde{H}-m=\sum_{\mathbf{k}}(\omega_{\mathbf{k}}+\epsilon_{\mathbf{k}}-m)a_{\mathbf{k}}{}^*a_{\mathbf{k}}+gV^{-1/2}\sum_{\mathbf{k}}(2\omega_{\mathbf{k}})^{-1/2}(\epsilon_{\mathbf{k}}m)^{-1}(a_{\mathbf{k}}{}^*+a_{\mathbf{k}})+\cdots. \tag{43}$$

If one keeps only the explicitly indicated terms of the expansion shown in (43), one obtains a *static-source meson theory*, the solution of which is well known for scalar particles. There is nothing "nonrelativistic" about this approximation; the validity of leaving off the higher terms in (43) when calculating, e.g., the (logarithmically divergent) self-energy is essentially independent of the cutoff.

[6] This reduced Hamiltonian is the prototype of all such obtained in interacting field theories after eliminating translational invariance. It may be considered as showing *exactly* the effect of *recoil* and may be useful as the basis of approximations other than the one to be described.

V. APPLICATION TO THE MESON-NUCLEON SYSTEM

When one applies the same steps to the usual pseudoscalar meson theory of nuclear interactions as was done in the previous section, expressions very similar to those used as the starting point of Chew-Low theory result.[7] In addition, higher order approximations to the effective Hamiltonian are easily obtained from the original Hamiltonian. The procedure and results will be outlined in this section.

One starts from the Hamiltonian

$$H=H_{0\phi}+H_{0\psi}+ig\int\bar{\psi}\gamma_5\boldsymbol{\phi}\cdot\boldsymbol{\tau}\psi d\mathbf{x}. \tag{44}$$

$H_{0\phi}$ and $H_{0\psi}$ are the free-field Hamiltonians of the pion and nucleon fields, respectively, $\boldsymbol{\phi}$ is the isovector pion field operator, $\boldsymbol{\tau}$ are the Pauli matrices operative in isospace, and ψ the nucleon field. The true problem is approximated for the purposes of calculating low-lying states by confining the states to the one-nucleon subspace.[4] Finally, using the conservation of momentum laws valid for the system, the nucleon momenta are culled from the problem leaving an effective problem in a reduced Hilbert space describing the meson field and the nucleon spin and isospin. Since the procedure is exactly the same as in the scalar case presented in IV, we simply write the result,

$$\tilde{H}=H_{0\phi}+\epsilon+\frac{i}{2}g\epsilon^{-1/2}[(\epsilon+m)^{1/2}\mathbf{P}_\phi\cdot\boldsymbol{\sigma}\boldsymbol{\phi}(0)\cdot\boldsymbol{\tau}(\epsilon+m)^{-1/2}-\text{Hermitian conjugate}]\epsilon^{-1/2}, \tag{45}$$

where now

$$\epsilon\equiv(\mathbf{P}_\phi{}^2+m^2)^{\frac{1}{2}},$$

$\boldsymbol{\sigma}$ are the Pauli matrices operative in ordinary spin space, and $\mathbf{P}_\phi$ is the total momentum operator of the pion field.

The expression (45) may be expanded in normal ordered form. Static-source theory again results from keeping the lowest nontrivial term

$$\tilde{H}-m\approx\sum_{\mathbf{k}}(\omega_{\mathbf{k}}+\epsilon_{\mathbf{k}}-m)a_{\mathbf{k}}{}^*a_{\mathbf{k}}+(ig/4m)V^{-1/2}\sum_{\mathbf{k}}(\epsilon_{\mathbf{k}}+m)^{1/2}\times(\omega_{\mathbf{k}}\epsilon_{\mathbf{k}})^{-1/2}\mathbf{k}\cdot\boldsymbol{\sigma}[a^*\cdot\tau-a\cdot\tau]. \tag{46}$$

The approximation of (45) by (46) is not essentially nonrelativistic.

APPENDIX: ON THE DEVELOPMENT IN NORMAL ORDERED FORM OF CERTAIN OPERATORS

Let

$$\{P_1,P_2,\cdots\}\equiv P \tag{47}$$

[7] See, for example, S. S. Schweber, *Relativistic Quantum Field Theory* (Row, Peterson, and Company, White Plains, N. Y., 1961), p. 377, Eq. 193b.

be a set of operators of the form

$$P=\sum_k a_k{}^* a_k p_k\,, \tag{48}$$

where the index k stands for the set of *all* one particle indices of a field (e.g., for a vector boson it includes vector as well as momentum indices) and where p_k is an arbitrary c-number function of the k. Let $f(P)$ be any function of the P, we wish to find the coefficients of the expansion

$$f(P)=\sum_{n=0}^{\infty}\sum_{k_i}(n!)^{-1}f_{k_1\cdots k_n}{}^{(n)}a_{k_1}{}^*\cdots a_{k_n}{}^* a_{k_1}\cdots a_{k_n}. \tag{49}$$

First note that

$$\langle 0|\,a_k\cdots a_{k_n}f(P)a_{k_1}{}^*\cdots a_{k_n}{}^*|\,0\rangle = f(p_{k_1}+\cdots+p_{k_n}) \tag{50}$$

when all k_i differ. Alternatively,

$$\begin{aligned}&\langle 0|\,a_{k_1}\cdots a_{k_n}f(P)a_{k_1}{}^*\cdots a_{k_n}{}^*|\,0\rangle\\ &=\sum_{m=0}^{\infty}\sum_{j_i}(m!)^{-1}f_{j_1\cdots j_m}{}^{(m)}\\ &\quad\times\langle 0|\,a_{k_1}\cdots a_{k_n}a_{j_1}{}^*\cdots a^*{}_{j_m}a_{j_1}\cdots a_{j_m}a^*{}_{k_1}\cdots a^*{}_{k_n}|\,0\rangle\\ &=\sum_{m=0}^{n}\underset{\substack{\alpha_1\cdots\alpha_m\\ 1\cdots n}}{\text{Comb}}\, f_{k\alpha_1\cdots k\alpha_m}{}^{(m)}\,,\end{aligned} \tag{51}$$

where Comb means the sum over all $(n!/m!)$ assignments of the α to the values $1\cdots n$ not differing by more permutations. Equations (50) and (51) together imply

$$\sum_{m=0}^{n}\underset{\substack{\alpha_1\cdots\alpha_m\\ 1\cdots n}}{\text{Comb}}\, f_{k\alpha_1\cdots k\alpha_m}{}^{(m)}=f(p_{k_1}+\cdots+p_{k_n}) \tag{52}$$

and the problem is now to invert this relationship. This is most easily done by defining the linear operators O_k

$$O_{k_1}O_{k_2}\cdots O_{k_n}f^{(0)}=f_{k_1}{}^{(n)}\cdots{}_{k_n} \tag{53}$$

$$(O_{k_1}+O_{k_2})f^{(0)}=f_{k_1}{}^{(1)}+f_{k_2}{}^{(1)} \tag{54}$$

then obviously

$$[O_{k_1},O_{k_2}]=0. \tag{55}$$

With these, relation (52) can be written

$$(1+O_{k_1})(1+O_{k_2})\cdots(1+O_{k_n})f^{(0)} =f(p_{k_1}+\cdots+p_{k_n})\,. \tag{56}$$

Now using the same reasoning which leads from (52) to (56) one sees

$$\begin{aligned}&(1+O_{k_1}-1)(1+O_{k_2}-1)\cdots(1+O_{k_n}-1)f^{(0)}\\ &=\sum_{m=0}^{n}(-)^{n-m}\underset{\substack{\alpha_1\cdots\alpha_m\\ 1\cdots n}}{\text{Comb}}(1+O_{k\alpha_1})\cdots(1+O_{k\alpha_m})f^{(0)}\\ &=\sum_{m=0}^{n}(-)^{n-m}\underset{\substack{\alpha_1\cdots\alpha_m\\ 1\cdots n}}{\text{Comb}}\, f(p_{k\alpha_1}+\cdots+p_{k\alpha_m})=f_{k_1\cdots k_n}{}^{(n)}\,,\end{aligned} \tag{57}$$

which is the inverted formula and displays a beautiful symmetry between the sets of functions $f(k_1+\cdots+k_n)$ and $f_{k_1}{}^{(n)}\cdots{}_{k_n}$.

In order to use Eq. (57) to develop expressions of the form shown in Eq. (42) one merely notes that

$$f(P)a_\mu{}^* g(P)=a_k{}^* f(P+p_k)g(P) \tag{58}$$

and proceeds using (57).

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